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# Cosuns in $l^p(n)$, $1 \leqslant p < \infty$

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## 1. INTRODUCTION

Referring to a paper of Franchetti and Furi [14] the concept of best coapproximation in normed linear spaces was introduced by Papini and Singer [20] in 1979 as a counterpart to the well established best approximation. If $K$ is a nonempty subset and $x$ an element of a normed linear space $X$, then in best approximation, those elements $k \in K$ (if any) are considered for which

$$\|x-k\| \leqslant \|x-k'\| \qquad \forall k' \in K,$$

while in best coapproximation, the interest is in those $k \in K$ (if any) satisfying

$$\|k-k'\| \leqslant \|x-k'\| \qquad \forall k' \in K. \tag{1.1}$$

Among others, best coapproximation seems to be an adequate tool in describing certain phenomena relevant in the study of contractive projections and sunny retractions which, in turn, are related to the theory of fixed points of contractive and quasi-contractive mappings (cf. for instance [2, 9, 10, 11, 12]). In particular, what is called a "cosun" in the setting of best coapproximation was used by Browder [9] in his study of approximants to fixed points of contractive mappings to generalize the "sun" property of closed sets in Hilbert spaces to more general Banach spaces. For precise definitions see Section 2 below.

In his thesis L. Hetzelt [16] intensively studied best coapproximation in finite dimensional normed linear spaces, in particular, metrical properties for cosuns in the normed plane as well as in strictly convex spaces. A cosun in an arbitrary normed linear space always is a closed norm-convex set. The converse—a closed norm-convex set is a cosun—is valid in the normed plane (cf. [15, 17]) as well as in Hilbert spaces and characterizes Hilbert



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spaces among the Banach spaces of dimension greater than two [16]. Therefore, in normed linear spaces of three and higher dimensions, in general, additional properties are necessary for characterizing cosuns. A contribution in this direction is a result of L. Hetzelt [16] for $n$-dimensional $l^p$-spaces, $1<p<\infty$, $p\neq 2$, related to the concept of $H$-convexity (cf. [8]).

> A subset of $\mathbb{R}^n$ is a cosun in $l^p(n)$, $1<p<\infty$, $p\neq 2$, if and only if it is the intersection of a family of closed half spaces whose normal vectors have at most two coordinates different from zero. (1.2)

This is an extension of a result of Bohnenblust [7] characterizing those linear subspaces of $l^p(n)$, $1<p<\infty$, $p\neq 2$, which are the range of a contractive linear projection.

In nonstrictly convex $l^p(n)$-spaces characterizations of cosuns seem to be unknown. It is the aim of this paper to describe cosuns in $l^1(n)$. Furthermore, we give a unified approach to cosuns in $l^p(n)$ for the two cases: $1<p<\infty$, $p\neq 2$ and $p=1$. It turns out that a kind of cylinder set may be regarded as fundamental for cosuns in $l^p(n)$, $1\leqslant p<\infty$, $p\neq 2$. By a cylinder set in $\mathbb{R}^n$ we mean a product of the form

$$\{(z_1, ..., z_n)\in\mathbb{R}^n;\ (z_i, z_j)\in B\}, \tag{1.3}$$

where $i, j\in\{1, ..., n\}$, $i<j$, and $B\subset\mathbb{R}^2$. We characterize the cosuns in $l^p(n)$, $1\leqslant p<\infty$, $p\neq 2$, as those norm-convex sets which are intersections of closed cylinder sets (cf. Theorem 4.12). The proof is established by an inductive argument and if $1<p<\infty$, $p\neq 2$ provides an alternate to the procedure in [16].

For $p=1$ we obtain a characterization of cosuns that may be considered as a direct analogue to (1.2). Indeed, the class of closed half spaces whose normal vectors have at most two coordinates different from zero is replaced by a class of norm-convex sets in $l^1(n)$ that have a quite simple structure, namely the so-called "angular spaces" whose both outer normal vectors lie in a quadrant of a two-dimensional coordinate plane. (By an angular space we mean a proper connected subset of $\mathbb{R}^n$ that is the union of two closed half spaces.) Besides, a condition is needed that guarantees that the intersection of a family of such norm-convex angular spaces of $l^1(n)$ is itself norm-convex (cf. Theorem 5.3).

Our investigations essentially depend on the fact that in the spaces under discussion, each existence set of best coapproximation is a cosun. The latter was proved by Hetzelt [17] even for general finite dimensional strictly convex spaces. We further extend this result to arbitrary finite dimensional spaces in Section 3 of this paper. Previously, in Section 2, some preliminaries are provided. In Section 4 we characterize cosuns in $l^p(n)$, $1 \leqslant p < \infty$, $p \neq 2$, by cylinder sets and in Section 5 by angular spaces.

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## 2. Preliminaries

Let $X$ be a finite dimensional real normed linear space with norm $\|\cdot\|$; in case $X = l^p(n)$ we also write $\|\cdot\|_p$ or $\|\cdot\|_{l^p(n)}$ for the norm. The semi-inner product $\langle \cdot, \cdot \rangle_s$ on $X \times X$ is defined by the right-hand Gâteaux derivative of $\|\cdot\|^2/2$

$$\langle x, y \rangle_s := \lim_{t \to 0+} \frac{\|y + tx\|^2 - \|y\|^2}{2t}, \qquad (x, y) \in X \times X.$$

For $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in l^p(n)$, $1 \leqslant p < \infty$, $y \neq 0$, there holds

$$\langle x, y \rangle_s = \begin{cases} \|y\|_1 \sum\limits_{\substack{i=1 \\ y_i \neq 0}}^{n} x_i \operatorname{sgn} y_i + \|y\|_1 \sum\limits_{\substack{i=1 \\ y_i = 0}}^{n} |x_i|, & p = 1 \\ \sum\limits_{i=1}^{n} x_i |y_i|^{p-1} \operatorname{sgn} y_i / \|y\|_p^{p-2}, & 1 < p < \infty. \end{cases}$$

For a subset $K$ of $X$, $\bar{K}$, $\mathring{K}$, and $\partial K$ denote its closure, interior, and boundary, respectively; moreover, $\operatorname{co} K$, $\operatorname{lin} K$, and $\operatorname{aff} K$ stand for its convex, linear, and affine hull, respectively. If $K$ is a subspace, $\dim K$ denotes its dimension.

We now introduce a special case of metrical convexity. If $x, y \in X$, $x \neq y$, we say that a point $z \in X$ is (metrically) *between* $x$ and $y$ or a *between-point* of $x$ and $y$, if $z \neq x, y$ and

$$\|x - z\| + \|z - y\| = \|x - y\|.$$

In a strictly convex space the set of points between $x$ and $y$ is the line segment $\{\alpha x + (1 - \alpha) y; 0 < \alpha < 1\}$. In case $X = l^1(n)$, the set of points between $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ is given by

$$\{z = (z_1, ..., z_n); \min\{x_i, y_i\} \leqslant z_i \leqslant \max\{x_i, y_i\} \ \forall i \in \mathbb{N}_n\} \backslash \{x, y\},$$

where $\mathbb{N}_n := \{1, ..., n\}$. A set $K \subset X$ is called *norm-convex* (also: $l^p(n)$ – convex, if $X = l^p(n)$), if for any two distinct points $k', k'' \in K$ there exists at least one between-point $k \in K$. By Menger, norm-convexity of closed sets can be characterized as follows (cf., e.g., [6]):

> A closed subset $K$ of $X$ is norm-convex if and only if any two distinct points $k', k'' \in K$ can be joined by a rectifiable curve of length $\|k'-k''\|$ entirely contained in $K$.

In a strictly convex space a closed norm-convex set is always convex.

For a nonempty set $K \subset X$ and an element $x \in X$ let

$$B_K(x) := \{z \in X;\ \|z-k\| \leqslant \|x-k\|\ \forall k \in K\}. \tag{2.1}$$

The *metric coprojection* from $X$ to $K$ is the set-valued mapping $R_K\colon X \to 2^X$, defined by

$$R_K(x) := B_K(x) \cap K.$$

If $K$ is closed, then $R_K$ is upper semi-continuous and compact-valued. For each $x \in X$, any $k \in R_K(x)$ is called an *element of best coapproximation* of $x$ in $K$. It is convenient to identify a set-valued mapping with its graph in $X \times X$; thus for instance, we also write $(x, k) \in R_K$ instead of $k \in R_K(x)$. $K$ is called an *existence set of best coapproximation*, if $R_K(x) \neq \varnothing$ for each $x \in X$, and a *cochebyshev set*, if $R_K(x)$ is a singleton for each $x \in X$.

If $K$ is an existence set of best coapproximation, then $K$ as well as $R_K(x)$, for each $x \in X$, are closed norm-convex sets, thus convex, if $X$ is strictly convex. In a strictly convex space, $K$ is an existence set of best coapproximation if and only if it is an optimal set in the sense of Beauzamy–Maurey [3, Prop. III.1.]. The connection between existence sets of best coapproximation and optimal sets in nonstrictly convex spaces was completely described by Hetzelt in his thesis (cf. [17]). We summarize a few results on existence sets of best coapproximation in strictly convex spaces to which we refer later in case $X = l^p(n)$, $1 < p < \infty$ (cf. [3, 16]).

If $X$ is strictly convex, then the following hold:

(i) The intersection of a family of existence sets of best coapproximation is an existence set of best coapproximation.

(ii) Let $K \subset X$ be an existence set of best coapproximation. If $\mathring{K} \neq \varnothing$, then every closed half space determined by a supporting hyperplane to $K$ at a smooth boundary point of $K$ is an existence set of best coapproximation. If $\mathring{K} = \varnothing$, then aff $K$ is an existence set of best coapproximation.

In a smooth space $X$, an existence set of best coapproximation that is a linear subspace is even a cochebyshev set and, moreover, the range of a unique contractive linear projection. If $X$ is $l^p(n)$, $1 < p < \infty$, the contractive linear retracts and their translates are the only cochebyshev sets of the space (cf. [5, 22]).

Concerning the definition of a cosun, let us introduce, for $x \in X$, the set

$$G_x := \{z \in X, \langle x, -z \rangle_s \geqslant 0\}.$$

The complement of $G_x$ with respect to $X$ is the "dual" analogue to the cone of decrease of $x$ with vertex at 0 which occurs in the setting of best approximation in connection with suns. If $K$ is a nonempty subset of $X$, the *approximation region* between $x$ and $K$ is the set

$$\begin{aligned} A_K(x) &:= \{z \in X; K \subset z + G_{x-z}\} \\ &= \{z \in X; \|z-k\| \leqslant \|z + \lambda(x-z) - k\| \ \forall k \in K, \forall \lambda > 0\}. \end{aligned}$$

This concept originating from F. Browder [9] was used by R. E. Bruck [10] for studying contractive retractions. The set-valued analogue of the so-called "orthogonal retractions" in [10] is the mapping $R'_K\colon X \to 2^K$ defined by

$$R'_K(x) := A_K(x) \cap K$$

(cf. [19, 20]). $K$ is called a *cosun* if $R'_K(x) \neq \varnothing$ for each $x \in X$. Since $R'_K \subset R_K$, a cosun is an existence set of best coapproximation. Moreover, $K$ is a cosun if and only if for each $x \in X$, there is at least one $k \in R_K(x)$ which is an element of best coapproximation of each point on the ray through $x$ originating from $k$. This illustrates the relations between cosuns and suns (see [21]). Note that a cosun $K$ can be represented in the form

$$K = \bigcap_{(x,k) \in R'_K} (k + G_{x-k})$$

which will be used especially for our investigations in the space $l^1(n)$.

## 3. The "Cosun" Property of Existence Sets

In [17] L. Hetzelt gave the following description of cosuns carrying over a result of H. Berens [4] on suns in finite dimensional spaces to the best coapproximation.

> A subset $K$ of a finite dimensional normed linear space $X$ is a cosun if and only if the map $\lambda I + (1-\lambda) R_K$ is surjective for all $\lambda > 1$. (3.1)

In [17] the surjectivity condition is verified for existence sets of best coapproximation in strictly convex spaces by a theorem from combinatorial topology on set-valued upper semi-continuous, closed- and convex-valued mappings with the additional property of being "outward directed" (see

[18]). Thus, in a strictly convex space an existence set of best coapproximation is a cosun. We extend this result to nonstrictly convex spaces by replacing the metric coprojection $R_K$ in (3.1) by its convex-valued extension

$$\Phi_K: X \to 2^X, \qquad \Phi_K(x) := \operatorname{co} R_K(x).$$

Cf. Proposition 3.2 below with the different situation in the setting of best approximation as described by H. Berens [4, Satz 2].

PROPOSITION 3.2. *A subset $K$ of a finite dimensional normed linear space $X$ is a cosun if and only if $\lambda I + (1-\lambda)\Phi_K$ is surjective for all $\lambda > 1$.*

*Proof.* For $K \subset X$ let us consider the compact- and convex-valued map $B_K: X \to 2^X$ defined by (2.1) satisfying the following easily verified property.

If $x \in X$, $z \in B_K(x)$, and $\lambda \geqslant 1$, then $z \in B_K(z + \lambda(x-z))$. Thus $z \in A_K(x)$ if and only if $z \in B_K(z + \lambda(x-z))$ for $\forall \lambda \in (0, 1)$.

Let $x \in X \backslash K$. We will show that the surjectivity condition implies $R'_K(x) \neq \varnothing$. For every $m \in \mathbb{N}$, $m \geqslant 2$, there is some $(x^m, y^m) \in \Phi_K$ such that $x = mx^m + (1-m)\,y^m$. As $y^m \in B_K(x^m)$, we have, by the above remark,

$$y^m \in B_K(x^m + \delta(x - x^m)) \qquad \forall m \geqslant 2,\ \forall \delta \in [0, 1]. \tag{3.3}$$

For $\delta = 1$ this implies that $(y^m)_{m \in \mathbb{N}}$ is a bounded sequence in $X$ which may be supposed convergent without loss of generality. Then,

$$\lim_{m \to \infty} x^m = \lim_{m \to \infty} y^m =: y.$$

The surjectivity condition implies $K$ to be closed, then $\Phi_K$ is upper semicontinuous and compact-valued, and thus $y \in \Phi_K(y)$. This means, $y$ is a convex combination of elements of $R_K(y)$, say $y = \sum_{i=1}^{l} \alpha^i k^i$, where $l \in \mathbb{N}$, $k^i \in R_K(y)$, $\alpha^i \geqslant 0$ $(i = 1, ..., l)$, and $\sum_{i=1}^{l} \alpha^i = 1$. Then, for every $j \in \{1, ..., l\}$,

$$\|y - k^j\| = \left\| \sum_{i \neq j} \alpha^i (k^i - k^j) \right\| \leqslant (1 - \alpha^j)\,\|y - k^j\|,$$

hence $\alpha^j \|y - k^j\| = 0$, from which we conclude that $y \in K$. It follows from (3.3), as $m \to \infty$,

$$y \in B_K(y + \delta(x - y)) \qquad \forall \delta \in [0, 1],$$

thus $y \in A_K(x) \cap K = R'_K(x)$.

We now proceed analogously to [17]. If $K$ is an existence set of best coapproximation and $x \in X \backslash K$, then for every $z \in \partial B_K(x)$, the set $\Phi_K(z)$ lies in the support cone to $B_K(x)$ at $z$. Thus, for every $\lambda > 1$ the mapping $\lambda I + (1-\lambda)\Phi_K$ restricted to the compact and convex set $B_K(x)$ is "outward directed." Moreover, it is upper semi-continuous and assigns a nonempty compact and convex set to each element of $B_K(x)$. By the surjectivity theorem in [18, Theorem 3.14, p. 47], it follows

$$B_K(x) \subset (\lambda I + (1-\lambda)\,\Phi_K)\,B_K(x).$$

Thus the mappings $\lambda I + (1-\lambda)\,\Phi_K$ are surjective for all $\lambda > 1$.

THEOREM 3.4. *In a finite dimensional normed linear space, an existence set of best coapproximation is a cosun.*

## 4. CYLINDER SETS

We first introduce some notation. For $n \in \mathbb{N}$ let $\mathbb{N}_n := \{1, ..., n\}$. $e^i$ denotes the $i$th canonical unit vector of $\mathbb{R}^n$. We define the following orthogonal projections. For $i \in \mathbb{N}_n$ let

$$\tau_i\colon \mathbb{R}^n \to \mathbb{R}^{n-1}, \qquad \tau_i(z) := (z_1, ..., z_{i-1}, z_{i+1}, ..., z_n), \qquad \text{if} \quad z = (z_1, ..., z_n),$$

$$\pi_i\colon \mathbb{R}^n \to \mathbb{R}, \qquad \pi_i(z) := z_i.$$

For $i, j \in \mathbb{N}_n$, $i < j$, let

$$\pi_{ij}\colon \mathbb{R}^n \to \mathbb{R}^2, \qquad \pi_{ij}(z) := (z_i, z_j).$$

If necessary for distinction, we provide the signs for these projections with a superscript $n$, e.g., $\tau_i^n$ instead of $\tau_i$, to indicate that $\mathbb{R}^n$ is the domain of $\tau_i$.

If $i, j \in \mathbb{N}_n$ such that $i < j$, then the 2-dimensional subspace of $\mathbb{R}^n$

$$\{z \in \mathbb{R}^n;\, \pi_r(z) = 0 \ \forall r \in \mathbb{N}_n \backslash \{i, j\}\}$$

is called the *$(i, j)$-coordinate plane*. It is decomposed into four quadrants according to the partition of $\mathbb{R}^2$. We always assume a quadrant to be a closed set. A set $C \subset \mathbb{R}^n$ is called a *cylinder set*, if there exist $i, j \in \mathbb{N}_n$, $i < j$, and a set $B \subset \mathbb{R}^2$ such that

$$C = \{z \in \mathbb{R}^n;\, \pi_{ij}(z) \in B\}. \tag{4.1}$$

The set

$$\{z \in \mathbb{R}^n;\, \pi_{ij}(z) \in B,\ \pi_r(z) = 0 \ \forall r \in \mathbb{N}_n \backslash \{i, j\}\}$$

is then called the *base* of $C$ *in the $(i, j)$-coordinate plane*. Trivially, there exist cylinder sets with bases in different 2-dimensional coordinate planes.

In this section we deal with cylinder sets that are cosuns in $l^p(n)$, $1 \leqslant p < \infty$, $p \neq 2$. They can be easily characterized as follows.

If $1 \leqslant p < \infty$, $p \neq 2$, and $C$ is a cylinder set of the form (4.1), then the following four assertions are equivalent:

(i) $C$ is a cosun in $l^p(n)$.

(ii) $C$ is closed and $l^p(n)$-convex.

(iii) $B$ is closed and $l^p(2)$-convex.

(iv) $B$ is a cosun in $l^p(2)$.

Moreover, for every $x \in \mathbb{R}^n$

$$\{z; \pi_{ij}(z) \in R_B(\pi_{ij}(x)), \pi_r(z-x) = 0 \ \forall r \in \mathbb{N}_n \backslash \{i, j\}\} \subset R_C(x).$$

Clearly, if $p > 1$ the intersection of a family of closed $l^p(n)$-convex—and thus convex—cylinder sets is always closed and ($l^p(n)$-) convex. The corresponding statement in case $p = 1$ is not valid in general, as can be seen already by simple examples for $n = 3$. Anyhow, the following lemma holds which, for convenience, is stated also for $p > 1$.

LEMMA 4.2. *If a nonempty set $K \subset \mathbb{R}^n$ is the intersection of a family of closed cylinder sets in $\mathbb{R}^n$, then $K$ can be represented in the form*

$$K = \bigcap_{1 \leqslant i < j \leqslant n} \{z \in \mathbb{R}^n; \pi_{ij}(z) \in \overline{\pi_{ij}(K)}\},$$

*and for $1 \leqslant p < \infty$, $p \neq 2$, $K$ is $l^p(n)$-convex if and only if $\overline{\pi_{ij}(K)}$ is $l^p(2)$-convex for all $(i, j) \in \mathbb{N}_n \times \mathbb{N}_n$ with $i < j$.*

*Proof.* We prove the assertion on norm-convexity in case $p = 1$. Let $K$ be $l^1(n)$-convex and let $(i, j) \in \mathbb{N}_n \times \mathbb{N}_n$, $i < j$. Then $\pi_{ij}(K)$ has the property that any two distinct points $(a_i, a_j)$, $(b_i, b_j) \in \pi_{ij}(K)$ can be joined by a rectifiable curve of length $|a_i - b_i| + |a_j - b_j|$ lying entirely in $\pi_{ij}(K)$. Then also the closure $\overline{\pi_{ij}(K)}$ has this property and thus is $l^1(2)$-convex.

Conversely, let all $\overline{\pi_{ij}(K)}$ be $l^1(2)$-convex and let $a, b$ be two distinct points in $K$. If, e.g., $(a_1, a_2) \neq (b_1, b_2)$ then $\overline{\pi_{12}(K)}$ contains a between-point $(c_1, c_2)$ of $(a_1, a_2)$ and $(b_1, b_2)$. This implies the existence of a sequence $(k^m)$ in $K$ such that $\lim_{m \to \infty} \pi_{12}(k^m) = (c_1, c_2)$ and for every $r = 3, ..., n$, the sequence of numbers $(k_r^m)$ has a finite or infinite limit. For $r \in \mathbb{N}_n$ put $k_r := \lim_{m \to \infty} k_r^m$ and for $r = 3, ..., n$ put

$$c_r := \begin{cases} \max\{a_r, b_r\} & \max\{a_r, b_r\} < k_r \leqslant \infty \\ \min\{a_r, b_r\} \quad \text{if} & -\infty \leqslant k_r < \min\{a_r, b_r\} \\ k_r & \min\{a_r, b_r\} \leqslant k_r \leqslant \max\{a_r, b_r\}. \end{cases} \tag{4.3}$$

Then $c := (c_1, ..., c_n)$ is a point between $a$ and $b$. Since for every

$(i, j) \in \mathbb{N}_n \times \mathbb{N}_n$, $i<j$, $\overline{\pi_{ij}(K)}$ is a closed $l^1(2)$-convex set containing the elements $(a_i, a_j)$, $(b_i, b_j)$, and $(k_i^m, k_j^m)$ $(m \in \mathbb{N})$, we conclude that $(c_i, c_j) \in \overline{\pi_{ij}(K)}$ and hence $c \in K$. Indeed, let $(i, j) \in \mathbb{N}_n \times \mathbb{N}_n$ such that $i<j$ and $(i, j) \neq (1, 2)$ and let, e.g., $a_i < b_i$ and $a_j < b_j$. To illustrate we consider the following three cases:

(i) $k_i \in (b_i, \infty]$, $k_j \in (b_j, \infty]$,

(ii) $k_i \in [-\infty, a_i)$, $k_j \in (b_j, \infty]$,

(iii) $k_i \in [a_i, b_i]$, $k_j \in (b_j, \infty]$.

In case (i), $(c_i, c_j) = (b_i, b_j) \in \overline{\pi_{ij}(K)}$. In case (ii), there exists a number $m_0 \in \mathbb{N}$ such that $k_i^m < a_i$ and $k_j^m > b_j$ for all integers $m \geqslant m_0$. Then for each $m \geqslant m_0$, we can find a $u_i^m \in \mathbb{R}$ such that $(u_i^m, b_j)$ is between $(k_i^m, k_j^m)$ and $(a_i, a_j)$ and belongs to $\overline{\pi_{ij}(K)}$. As $u_i^m \leqslant a_i < b_i$, also $(c_i, c_j) = (a_i, b_j)$ is contained in $\overline{\pi_{ij}(K)}$. As well in case (iii), for all sufficiently large $m$ there exists an element $(u_i^m, b_j) \in \overline{\pi_{ij}(K)}$ which is a between-point of $(k_i^m, k_j^m)$ and $(a_i, a_j)$. As $(k_i^m)$ is a bounded sequence, also $(u_i^m)$ is bounded and hence has a convergent subsequence with limit, say $u_i$. As $(u_i, b_j) \in \overline{\pi_{ij}(K)}$ and $u_i \leqslant k_i \leqslant b_i$, also $(c_i, c_j) = (k_i, b_j)$ belongs to $\overline{\pi_{ij}(K)}$.

We now wish to show that a cosun $K$ in $l^p(n)$, $1 \leqslant p < \infty$, $p \neq 2$, can be represented as the intersection of a family of closed $l^p(n)$-convex cylinder sets. For this, we proceed by iteration, we first prove that, if $K$ is a cosun in $l^p(n)$, it admits the representation

$$K = \bigcap_{i=1}^{n} \{z \in \mathbb{R}^n;\ \tau_i(z) \in \overline{\tau_i(K)}\}; \tag{4.4}$$

then we show that $\overline{\tau_i(K)}$ is a cosun in $l^p(n-1)$. In order to verify (4.4) we treat the cases $p=1$ and $1<p<\infty$, $p \neq 2$, separately.

If $p=1$, a representation of the form (4.4) can be proved for the sets $G_x$ in $l^1(n)$. Recall that for every $x \in l^1(n)$

$$\begin{aligned} G_x &= \{z \in l^1(n);\ \langle x, -z \rangle_s \geqslant 0\} \\ &= \left\{(z_1, \ldots, z_n) \in \mathbb{R}^n;\ \sum_{\substack{i=1 \\ z_i \neq 0}}^{n} x_i \operatorname{sgn} z_i \leqslant \sum_{\substack{i=1 \\ z_i = 0}}^{n} |x_i| \right\}. \end{aligned}$$

LEMMA 4.5. *If* $x \in l^1(n)$, $n \geqslant 3$, *then*

$$G_x = \bigcap_{i=1}^{n} \{z \in l^1(n);\ \tau_i(z) \in \tau_i(G_x)\}.$$

*Proof.* Let $z \in l^1(n)$ such that $\tau_i(z) \in \tau_i(G_x)$ for each $i \in \mathbb{N}_n$. Then

$z - z_i e^i \in G_x$ for each $i \in \mathbb{N}_n$. Thus, if $z_i = 0$ for some $i \in \mathbb{N}_n$, then $z \in G_x$. If, however, $z_i \neq 0$ for each $i \in \mathbb{N}_n$, we have the inequalities

$$\sum_{j=1}^{n} x_j \operatorname{sgn} z_j \leqslant x_i \operatorname{sgn} z_i + |x_i| \qquad (i \in \mathbb{N}_n),$$

which, by addition, imply

$$(n-1) \sum_{i=1}^{n} x_i \operatorname{sgn} z_i \leqslant \sum_{i=1}^{n} |x_i|.$$

Supposing $\operatorname{sgn} z_i = \operatorname{sgn} x_i$ for each $i \in \mathbb{N}_n$ gives a contradiction. Indeed, on the one hand, $x \neq 0$ since $z_i \neq 0$ for each $i \in \mathbb{N}_n$, on the other hand, $(n-2) \sum_{i=1}^{n} |x_i| \leqslant 0$, hence $x = 0$. Therefore, there must be an $i \in \mathbb{N}_n$ such that $\operatorname{sgn} z_i \neq \operatorname{sgn} x_i$ and hence $x_i \operatorname{sgn} z_i + |x_i| = 0$ which implies $\sum_{j=1}^{n} x_j \operatorname{sgn} z_j \leqslant 0$. Thus, $z \in G_x$.

PROPOSITION 4.6. *If $K$ is a cosun in $l^1(n)$, $n \geqslant 3$, then*

$$K = \bigcap_{i=1}^{n} \{z \in l^1(n);\, \tau_i(z) \in \overline{\tau_i(K)}\}.$$

*Proof.* Recalling that

$$K = \bigcap_{(x,k) \in R'_K} (k + G_{x-k})$$

and using the representation formula from Lemma 4.5 we have

$$K = \bigcap_{i=1}^{n} \left\{ z \in l^1(n);\, \tau_i(z) \in \bigcap_{(x,k) \in R'_K} \tau_i(k + G_{x-k}) \right\},$$

where the sets $\tau_i(k + G_{x-k})$ are closed for all $i \in \mathbb{N}_n$ and all $(x,k) \in R'_K$. From this we get the formula stated in the proposition.

Observe that Lemma 4.5 is no longer valid in $l^p(n)$, $1 < p < \infty$, $p \neq 2$. To prove formula (4.4) in this case we use a typical convexity argument instead noting that in a strictly convex space a cosun is a convex set. Thus, (4.4) is first verified for linear hyperplanes that are cosuns. If $K \subset \mathbb{R}^n$ is a linear subspace of dimension $\leqslant (n-2)$, (4.4) is valid in any case even if $K$ is not a cosun. The general case then can be handled by these linear results and known theorems on convex existence sets of best coapproximation.

LEMMA 4.7. *If $K$ is a linear hyperplane that is a cosun in $l^p(n)$, $1 < p < \infty$, $p \neq 2$, $n \geqslant 3$, then there is an $i \in \mathbb{N}_n$ such that*

$$K = \{z \in \mathbb{R}^n;\ \tau_i(z) \in \tau_i(K)\}.$$

*Proof.* For brevity, we put $K_i := \{z \in \mathbb{R}^n; \tau_i(z) \in \tau_i(K)\}$. Note that $K = K_i$ if and only if $e^i \in K$. Suppose that $e^i \notin K$ for each $i \in \mathbb{N}_n$. Then for each such $i$ there exists a unique $k^i \in K$ and a unique $\lambda_i \in \mathbb{R} \backslash \{0\}$ such that $e^1 = k^i + \lambda_i e^i$ ($k^1 = 0$, $\lambda_1 = 1$). The vectors $e^1 - \lambda_i e^i$ ($i = 2, ..., n$) then form a basis for $K$. Since $K$ is a linear cosun in a smooth space, it is a cochebyshev set. Let $x \notin K$ and put $y := x - R_K(x)$. Then $\langle y, k \rangle_s = 0$ for each $k \in K$. Choosing $k := e^1 - \lambda_i e^i$ ($i = 2, ..., n$) we obtain $y_1 = y_i |\lambda_i|^{p-1} \operatorname{sgn} \lambda_i$ ($i \in \mathbb{N}_n$), and hence

$$\sum_{i=1}^{n} \left| \frac{k_i}{\lambda_i} \right|^{p-1} \operatorname{sgn} \frac{k_i}{\lambda_i} = 0 \qquad \forall k \in K. \tag{4.8}$$

Taking now the linear combination $k := \sum_{i=2}^{n} \alpha_i (e^1 - \lambda_i e^i)$ with positive $\alpha_i$, (4.8) leads to the equation

$$\sum_{i=2}^{n} \alpha_i^{p-1} = \left( \sum_{i=2}^{n} \alpha_i \right)^{p-1} \qquad (\alpha_i > 0)$$

which is false, if $1 < p < \infty$, $p \neq 2$, and $n \geqslant 3$. Thus there must be at least one $i \in \mathbb{N}_n$ such that $e^i \in K$.

E.g., in [3] it is shown that under the hypotheses of Lemma 4.7 the hyperplane $K$ even contains at least $(n-2)$ canonical unit vectors of $\mathbb{R}^n$. Of course, the proof of this stronger result requires a more sophisticated argument.

Before taking up the general case let us mention that the formula given in Lemma 4.7 also holds for cosuns that are closed half spaces.

PROPOSITION 4.9. *If $K$ is a cosun in $l^p(n)$, $1 < p < \infty$, $p \neq 2$, $n \geqslant 3$, then*

$$K = \bigcap_{i=1}^{n} \{z \in \mathbb{R}^n;\ \tau_i(z) \in \overline{\tau_i(K)}\}.$$

*Proof.* Since $l^p(n)$ is strictly convex for the $p$'s in question, $K$ is a convex set. If $\mathring{K} \neq \varnothing$, let $Q$ be the set of smooth boundary points of $K$, and for each $q \in Q$, let $H_q$ be the closed half space which is determined by the supporting hyperplane to $K$ at $q$ and contains $K$. Then $Q$ is dense in $\partial K$ and

$$K = \bigcap_{q \in Q} H_q.$$

As $H_q$ is an existence set of best coapproximation and thus a cosun, we have

$$H_q = \bigcap_{i=1}^{n} \{z \in \mathbb{R}^n; \tau_i(z) \in \tau_i(H_q)\}.$$

Therefore,

$$K = \bigcap_{i=1}^{n} \left\{ z \in \mathbb{R}^n; \tau_i(z) \in \bigcap_{q \in Q} \tau_i(H_q) \right\},$$

where for each $i \in \mathbb{N}_n$ and each $q \in Q$, $\tau_i(H_q)$ is either a closed half space of $\mathbb{R}^{n-1}$ or the entire space $\mathbb{R}^{n-1}$, in any case a closed set. Thus the stated formula follows in this case.

If $\mathring{K} = \emptyset$, we may assume, without loss of generality that $0 \in K$. Then $\operatorname{lin} K$ is a proper linear subspace of $\mathbb{R}^n$, and moreover, an existence set of best coapproximation, thus admitting a representation in the form given by (4.4). Suppose there exists an $x \in \mathbb{R}^n \backslash K$ such that $\tau_i(x) \in \overline{\tau_i(K)}$ for each $i \in \mathbb{N}_n$. Then $x \in \operatorname{lin} K$ and for each $i \in \mathbb{N}_n$ there is a sequence $(k^m)$ in $K$ such that $x = \lim_{m \to \infty}(k^m + (x_i - k_i^m)e^i)$ and the sequence $(k_i^m)$ of real numbers has a finite or infinite limit. In both cases, it follows that $e^i \in \operatorname{lin} K$. Indeed, if $\lim_{m \to \infty} k_i^m =: k_i \in \mathbb{R}$, then $\lim_{m \to \infty} k^m =: k \in K$ and $x_i \neq k_i$, since $x \notin K$, hence

$$e^i = (x_i - k_i)^{-1}(x - k) \in \operatorname{lin} K.$$

If $\lim_{m \to \infty} k_i^m = \pm\infty$, then

$$e^i = \lim_{m \to \infty} (k_i^m)^{-1} k^m \in \operatorname{lin} K.$$

Since $e^i$ belongs to $\operatorname{lin} K$ for each $i \in \mathbb{N}_n$ and, on the other hand, $\dim(\operatorname{lin} K) \leqslant n - 1$, we obtain a contradiction.

PROPOSITION 4.10. *Let $1 \leqslant p < \infty$, $p \neq 2$, and $n \geqslant 3$. If $K$ is a cosun in $l^p(n)$, then for each $i \in \mathbb{N}_n$, $\overline{\tau_i(K)}$ is a cosun in $l^p(n-1)$.*

*Proof.* We prove that $\overline{\tau_1(K)}$ is an existence set of best coapproximation. Let us recall that for all $x \in l^p(n)$, $R_K(x)$ is a compact $l^p(n)$-convex subset of $K$ and hence $\pi_1(R_K(x))$ is a compact interval of $\mathbb{R}$.

Let $(z_2, ..., z_n) \notin \overline{\tau_1(K)}$. Then for all $\lambda \in \mathbb{R}$, $z^\lambda := (\lambda, z_2, ..., z_n) \notin K$. If there exists a $\lambda \in \mathbb{R}$ such that $\lambda \in \pi_1(R_K(z^\lambda))$, then for each $k \in R_K(z^\lambda)$ satisfying $\lambda = \pi_1(k)$, we have $\tau_1(k) \in R_{\overline{\tau_1(K)}}(z_2, ..., z_n)$.

Suppose now that for all $\lambda \in \mathbb{R}$, $\lambda \notin \pi_1(R_K(z^\lambda))$. Then the sets

$$A_1 := \{\lambda \in \mathbb{R}; \pi_1(R_K(z^\lambda)) \subset (-\infty, \lambda)\}$$

$$A_2 := \{\lambda \in \mathbb{R}; \pi_1(R_K(z^\lambda)) \subset (\lambda, \infty)\}$$

are disjoint and have union $\mathbb{R}$. Moreover, since $R_k$ is upper semi-continuous, $\pi_1$ continuous, and $z^\lambda$ continuously depends on $\lambda$, $\Lambda_1$ and $\Lambda_2$ are open subsets of $\mathbb{R}$. Hence, $\mathbb{R}$ being connected, either $\Lambda_1=\varnothing$ or $\Lambda_2=\varnothing$. If, for instance, $\Lambda_2=\varnothing$, then there exists a $\lambda_0\in\mathbb{R}$ such that $\bigcup_{\lambda\leqslant\lambda_0}\tau_1(R_K(z^\lambda))$ is a bounded subset of $\mathbb{R}^{n-1}$.

Indeed, if we fix a point $k\in K$, then for each $\lambda\leqslant k_1=:\lambda_0$ and each $k^\lambda\in R_K(z^\lambda)$, we have

$$\|\tau_1(k^\lambda)-\tau_1(k)\|_p<\left(\sum_{j=2}^n|z_j-k_j|^p\right)^{1/p},$$

since $|\lambda-k_1|^p<|k_1^\lambda-k_1|^p$. Consequently, there is a decreasing sequence $(\lambda_m)$ in $\mathbb{R}$ and a sequence $(k^m)$ in $K$ such that $\lim_{m\to\infty}\lambda_m=-\infty$, $k^m\in R_K(z^{\lambda_m})$ for all $m\in\mathbb{N}$, and the sequence $(\tau_1(k^m))$ converges to some element $u\in\overline{\tau_1(K)}$. One easily checks that $u\in R_{\overline{\tau_1(K)}}(z_2, ..., z_n)$.

Using Propositions 4.6, 4.9, and 4.10 systematically we proceed as follows. If $K$ is a cosun in $l^p(n)$, $1\leqslant p<\infty$, $p\neq 2$, $n\geqslant 3$, then

$$K=\bigcap_{i_1=1}^n\{z\in\mathbb{R}^n;\tau_{i_1}^n(z)\in\overline{\tau_{i_1}^n(K)}\}$$

and $\overline{\tau_{i_1}^n(K)}$ is a cosun in $l^p(n-1)$, thus it can be expressed as

$$\overline{\tau_{i_1}^n(K)}=\bigcap_{i_2=1}^{n-1}\{y\in\mathbb{R}^{n-1};\tau_{i_2}^{n-1}(y)\in\overline{(\tau_{i_2}^{n-1}\circ\tau_{i_1}^n)(K)}\}.$$

Combining both formulas we obtain

$$K=\bigcap_{i_1=1}^n\bigcap_{i_2=1}^{n-1}\{z\in\mathbb{R}^n;(\tau_{i_2}^{n-1}\circ\tau_{i_1}^n)(z)\in\overline{(\tau_{i_2}^{n-1}\circ\tau_{i_1}^n)(K)}\},$$

where $\overline{(\tau_{i_2}^{n-1}\circ\tau_{i_1}^n)(K)}$ is a cosun in $l^p(n-2)$. By iteration, it follows that

$$K=\bigcap_{i_1=1}^n\cdots\bigcap_{i_{n-2}=1}^3\{z\in\mathbb{R}^n;(\tau_{i_{n-2}}^3\circ\circ\circ\tau_{i_1}^n)(z)\in\overline{(\tau_{i_{n-2}}^3\circ\circ\circ\tau_{i_1}^n)(K)}\},$$

and $\overline{(\tau_{i_{n-2}}^3\circ\circ\circ\tau_{i_1}^n)(K)}$ is a cosun in $l^p(2)$.

Since for every $(n-2)$tuple $(i_{n-2}, ..., i_1)\in\mathbb{N}_3\times\cdots\times\mathbb{N}_n$ there is a unique pair $(i, j)\in\mathbb{N}_n\times\mathbb{N}_n$ with $i<j$ such that

$$\tau_{i_{n-2}}^3\circ\circ\circ\tau_{i_1}^n=\pi_{ij},$$

we have, in fact, the formula

$$K=\bigcap_{1\leqslant i<j\leqslant n}\{z\in\mathbb{R}^n;\pi_{ij}(z)\in\overline{\pi_{ij}(K)}\}. \tag{4.11}$$

So far we have proved that a cosun in $l^p(n)$, $1 \leqslant p < \infty$, $p \neq 2$, $n \geqslant 3$, is the intersection of a family of closed $l^p(n)$-convex cylinder sets. The converse is true if $1 < p < \infty$, $p \neq 2$, as follows from a general result on existence sets of best coapproximation in strictly convex spaces cited in Section 2. Concerning the case $p = 1$, we already mentioned that the intersection of an arbitrary family of closed $l^1(n)$-convex cylinder sets need not be always $l^1(n)$-convex. If, however, it is so, it is even a cosun in $l^1(n)$, as may be seen by Theorem 4.12 below. Thus, in order to obtain a cosun from a given family of closed cylinder sets—with the assertion of Lemma 4.2 in mind—it is reasonable to make the assumption of norm-convexity rather to the intersection than to each single member of the family.

THEOREM 4.12. *Let $1 \leqslant p < \infty$, $p \neq 2$, $n \geqslant 3$. A nonempty subset of $\mathbb{R}^n$ is a cosun in $l^p(n)$ if and only if it is $l^p(n)$-convex and the intersection of a family of closed cylinder sets.*

*Proof.* It remains to prove the sufficiency of the assertion. Let $K$ be the intersection of a family of closed cylinder sets such that $K$ is $l^p(n)$-convex. Then by Lemma 4.2, $K$ can be expressed in the form (4.11) and for each $(i, j) \in \mathbb{N}_n \times \mathbb{N}_n$ with $i < j$, $\overline{\pi_{ij}(K)}$ is a closed $l^p(2)$-convex subset of $\mathbb{R}^2$ which implies that every cylinder set

$$K_{ij} := \{z \in \mathbb{R}^n;\ \pi_{ij}(z) \in \overline{\pi_{ij}(K)}\}$$

$(i, j) \in \mathbb{N}_n \times \mathbb{N}_n$, $i < j$, is a cosun in $l^p(n)$. We claim that also the intersection $K = \bigcap_{1 \leqslant i < j \leqslant n} K_{ij}$ is an existence set of best coapproximation. This is known for $p > 1$. The following argument primarily intended for $p = 1$ also applies to the other $p$'s under discussion.

Let $x \in \mathbb{R}^n \backslash K$. By [3] there exists an element $z \in B_K(x)$ which is minimal in the weak sense with respect to $K$, i.e., if $z' \in B_K(z)$, then $\|z' - k\| = \|z - k\|$ for each $k \in K$. We shall prove that $z$ must belong to $K$. For this fix $(i, j) \in \mathbb{N}_n \times \mathbb{N}_n$, $i < j$. There exists a $u = (u_1, ..., u_n) \in R_{K_{ij}}(z)$ such that $u_r = z_r$ for each $r \in \mathbb{N}_n \backslash \{i, j\}$. Since $z$ is a minimal point in the weak sense with respect to $K$, it follows that

$$\|u - k\|_{l^p(n)} = \|z - k\|_{l^p(n)} \qquad \forall k \in K$$

whence

$$\|(u_i, u_j) - (k_i, k_j)\|_{l^p(2)} = \|(z_i, z_j) - (k_i, k_j)\|_{l^p(2)} \qquad \forall (k_i, k_j) \in \overline{\pi_{ij}(K)}.$$

Taking $(k_i, k_j) = (u_i, u_j)$ this equation leads to $u_i = z_i$, $u_j = z_j$, thus $u = z$ implying $z \in K_{ij}$. Since this applies to each $(i, j) \in \mathbb{N}_n \times \mathbb{N}_n$ with $i < j$, we have $z \in K$ and hence $z \in R_K(x)$.

## 5. Angular Spaces

We now wish to characterize cosuns in $l^p(n)$ by special families of simply constructed $l^p(n)$-convex cylinder sets, in particular in case $p=1$.

Concerning $l^p(n)$, $1<p<\infty$, $p\neq 2$, $n\geqslant 3$, Theorem 4.12 reads: A nonempty subset of $\mathbb{R}^n$ is a cosun in $l^p(n)$ if and only if it is the intersection of a family of closed convex cylinder sets.

Before going on, let us introduce the following notation. Let $H$ be a closed half space of $\mathbb{R}^n$ and $H_0$ that translate of $H$ whose bounding hyperplane contains the origin. Then by the *$l^2(n)$-normal* of $H$, we mean the unique vector $u\in l^2(n)\backslash H_0$ with $\|u\|_2=1$ that is orthogonal to $\partial H_0$ with respect to the $l^2(n)$-norm.

Plainly, a proper subset of $\mathbb{R}^n$ is a closed convex cylinder set with base in the $(i, j)$-coordinate plane if and only if it is the intersection of a family of closed half spaces whose $l^2(n)$-normals have at most the $i$th and $j$th coordinate different from zero. Thus, the known characterization of cosuns in $l^p(n)$, $1<p<\infty$, $p\neq 2$, is just a corollary of Theorem 4.12.

Corollary 5.1 (*of Theorem* 4.12). *A proper subset of $\mathbb{R}^n$ is a cosun in $l^p(n)$, $1<p<\infty$, $p\neq 2$, $n\geqslant 3$, if and only if it is the intersection of a family of closed half spaces whose $l^2(n)$-normals have at most two nonzero coordinates.*

In case $p=1$ the class of closed half spaces has to be replaced by a wider class of $l^1$-convex sets. To this end we introduce the following definitions.

A connected proper subset $A\subset\mathbb{R}^n$ $(n\geqslant 2)$ is called an *angular space* if there exist two closed half spaces $H_1$ and $H_2$ such that $A=H_1\cup H_2$. We call $(H_1, H_2)$ a *pair of generating half spaces* of $A$. An angular space $A$ is a half space itself if and only if the $l^2(n)$-normals of a pair of generating half spaces of $A$ coincide. If an angular space $A$ is not a half space then there is exactly one pair of generating half spaces of $A$. Their bounding hyperplanes intersect in an $(n-2)$-dimensional affine subspace which we call the *edge* of $A$. In case $n=2$ we merely speak of the *vertex* of $A$. (If $A$ is a half space, an edge is not defined.) By the *$l^2(n)$-normals of an angular space $A$* we mean the uniquely determined $l^2(n)$-normals of its generating half spaces. (They coincide if and only if $A$ itself is a half space.)

Throughout this section we deal with $l^1$-convex angular spaces that are cylinder sets. For their characterization we have

Lemma 5.2. (i) *Given an angular space $A\subset\mathbb{R}^2$ that is not a half space, with $l^2(2)$-normals $(g_1, g_2)$ and $(h_1, h_2)$, then $A$ is $l^1(2)$-convex if and only if $g_1h_1\geqslant 0$ and $g_2h_2\geqslant 0$, i.e., the $l^2(2)$-normals of $A$ lie in some quadrant of $\mathbb{R}^2$.*

(ii) *If* $(i, j) \in \mathbb{N}_n \times \mathbb{N}_n$ *with* $i < j$, *then a proper subset* $C \subset \mathbb{R}^n$ *is a cylinder set with base in the* $(i, j)$*-coordinate plane and* $\pi_{ij}(C)$ *is an* $l^1(2)$*-convex angular space of* $\mathbb{R}^2$ *if and only if* $C$ *is an angular space of* $\mathbb{R}^n$, *whose* $l^2(n)$*-normals lie in some quadrant of the* $(i, j)$*-coordinate plane.*

Now, if $K$ is a cosun in $l^1(n)$, it takes the form

$$K = \bigcap_{1 \leqslant i < j \leqslant n} \; \bigcap_{(y, w) \in R'_{\overline{\pi_{ij}(K)}}} \{z \in \mathbb{R}^n; \pi_{ij}(z) \in w + G_{y-w}\}.$$

If $(i, j) \in \mathbb{N}_n \times \mathbb{N}_n$ with $i < j$ and $(y, w) \in R_{\overline{\pi_{ij}(K)}}$ we set, for brevity,

$$A(i, j, y, w) := \{z \in \mathbb{R}^n; \pi_{ij}(z) \in w + G_{y-w}\}.$$

For all $y = (y_1, y_2)$ and all $w = (w_1, w_2)$ in $\mathbb{R}^2$ with $y \neq w$, $w + G_{y-w}$ is an $l^2(2)$-convex angular space of $\mathbb{R}^2$ of the form

$$\{(z_1, z_2); \alpha z_1 \leqslant \alpha w_1 \text{ or } \beta z_2 \leqslant \beta w_2\},$$

where $\alpha, \beta \in \{-1, 0, 1\}$ and $|\alpha| + |\beta| \neq 0$; indeed, $\alpha = 0$ or $\beta = 0$ if $|y_1 - w_1| \neq |y_2 - w_2|$, and $\alpha \neq 0$, $\beta \neq 0$ if $|y_1 - w_1| = |y_2 - w_2|$. In the first case, $w + G_{y-w}$ is a half space of $\mathbb{R}^2$, in the latter it is an angular space which is not a half space, with $l^2(2)$-normals $(\alpha, 0)$ and $(0, \beta)$ and vertex at $w$.

Thus, for all $(i, j) \in \mathbb{N}_n \times \mathbb{N}_n$, $i < j$, and all $(y, w) \in R_{\overline{\pi_{ij}(K)}}$, $y \neq w$, the cylinder set $A(i, j, y, w)$ is an angular space whose $l^2(n)$-normals lie in some quadrant of the $(i, j)$-coordinate plane. If $A(i, j, y, w)$ is not a half space, the set $\{z \in \mathbb{R}^n; \pi_{ij}(z) = w\}$ is its edge whose distance from $K$ is zero, since $w \in \overline{\pi_{ij}(K)}$.

Thus, one part of the following theorem is proved.

THEOREM 5.3. *A proper subset of* $\mathbb{R}^n$ $(n \geqslant 2)$ *is a cosun in* $l^1(n)$ *if and only if it is the intersection of a family* $\mathfrak{A}$ *of angular spaces of* $\mathbb{R}^n$ *with the following properties. For each* $A \in \mathfrak{A}$:

(i) *The* $l^2(n)$*-normals of* $A$ *lie in some quadrant of a two-dimensional coordinate plane of* $\mathbb{R}^n$.

(ii) *If* $A$ *is not a half space its edge has distance zero from the intersection of the family* $\mathfrak{A}$.

*Proof.* Given a family $\mathfrak{A}$ of angular spaces of $\mathbb{R}^n$ which satisfy both conditions (i), (ii) above, let $K := \bigcap_{A \in \mathfrak{A}} A$. By (i), for every $A \in \mathfrak{A}$, there is a pair $(s, t) \in \mathbb{N}_n \times \mathbb{N}_n$, $s < t$, such that

$$A = \{z \in \mathbb{R}^n; \pi_{st}(z) \in \pi_{st}(A)\}$$

and $\pi_{st}(A)$ is an $l^1(2)$-convex angular space of $\mathbb{R}^2$. (If $A$ is not a half space, the pair $(s, t)$ is unique.) Thus, $K$ takes the form

$$K = \bigcap_{1 \leqslant i < j \leqslant n} \left\{ z \in \mathbb{R}^n ;\ \pi_{ij}(z) \in \bigcap_{A \in \mathfrak{A}} \pi_{ij}(A) \right\},$$

where $\pi_{ij}(A)$ is either an $l^1(2)$-convex angular space of $\mathbb{R}^2$ or the entire space $\mathbb{R}^2$. We wish to show that $K$ is $l^1(n)$-convex. Then, by Theorem 4.12, $K$ is a cosun in $l^1(n)$.

Let $a, b \in K$, $a \neq b$. We first assume

There exists an angular space $A_0 \in \mathfrak{A}$ which is not a half space, such that its edge does not contain the elements $a, b$, but a between-point of $a$ and $b$. (5.4)

Let us suppose that the base of $A_0$ lies in the $(1, 2)$-coordinate plane and let us denote the vertex of $\pi_{12}(A_0)$ by $(c_1, c_2)$. Then, by (5.4), $(c_1, c_2)$ is a between-point of $\pi_{12}(a)$ and $\pi_{12}(b)$, and, by assumption (ii), belongs to $\overline{\pi_{12}(K)}$. We now proceed as in the second part of the proof of Lemma 4.2. For $r \in \mathbb{N}_n \setminus \{1, 2\}$ we may define $c_r$ as in (4.3) such that $c := (c_1, ..., c_n)$ is a point between $a$ and $b$. Since for each $(i, j) \in \mathbb{N}_n \times \mathbb{N}_n$ with $i < j$ and each $A \in \mathfrak{A}$ the set $\pi_{ij}(A)$ is closed and $l^1(2)$-convex containing the elements $\pi_{ij}(a)$, $\pi_{ij}(b)$, and $\pi_{ij}(k^m)$ $(m \in \mathbb{N})$, it follows analogously to the procedure in the proof of Lemma 4.2 that $\pi_{ij}(c) \in \pi_{ij}(A)$ for all $(i, j) \in \mathbb{N}_n \times \mathbb{N}_n$, $i < j$, and all $A \in \mathfrak{A}$; hence $c \in K$.

If the case (5.4) does not apply, then for each $A \in \mathfrak{A}$, both points $a$ and $b$ lie in one of the generating half spaces of $A$. Thus the line segment joining $a$ and $b$ must be contained in each $A$ and, consequently, in $K$.

As may be seen from the "only if" part of the proof of Theorem 5.3 the class of angular spaces fitting for characterizing cosuns in $l^1(n)$ may be even further restricted.

COROLLARY 5.5. *A proper subset of $\mathbb{R}^n$ $(n \geqslant 2)$ is a cosun in $l^1(n)$ if and only if it is the intersection of a family $\mathfrak{A}$ of angular spaces whose $l^2(n)$-normals belong to the set of unit vectors $\{\pm e^i; i \in \mathbb{N}_n\}$ and whose edges (if any) have distance zero from the intersection of the family $\mathfrak{A}$.*

## 6. CONCLUDING REMARKS

It is plain that for $p_1, p_2 \in (1, \infty) \setminus \{2\}$ a subset of $\mathbb{R}^n$ is a cosun with respect to $l^{p_1}(n)$ if and only if it is a cosun with respect to $l^{p_2}(n)$. To compare this with the situation in $l^1(n)$ let us point out the following corollary of Theorem 4.12.

PROPOSITION 6.1. *For a proper subset $K$ of $\mathbb{R}^n$ the following assertion are equivalent:*

(i) *$K$ is a convex cosun in $l^1(n)$.*

(ii) *$K$ is the intersection of a family of closed convex cylinder sets.*

(iii) *$K$ is the intersection of a family of closed half spaces whose $l^2(n)$-normals have at most two nonzero coordinates.*

(iv) *$K$ is a cosun in $l^p(n)$ for some $p$, $1 < p < \infty$, $p \neq 2$.*

Thus, the class of cosuns in $l^1(n)$ contains all subsets of $\mathbb{R}^n$ that are cosuns with respect to $l^p(n)$, $1 < p < \infty$, $p \neq 2$; in addition there are just those cosuns of $l^1(n)$ that are nonconvex. From this it is also clear that the classes of linear cosuns in $l^p(n)$ coincide for all $p$, $1 \leqslant p < \infty$, $p \neq 2$. Hence, the following characterization of linear cosuns known in $l^p(n)$-spaces for $1 < p < \infty$, $p \neq 2$, also holds in $l^1(n)$.

PROPOSITION 6.2. *Let $1 \leqslant p < \infty$, $p \neq 2$. For a proper linear subspace $K$ of $\mathbb{R}^n$ the following assertions are equivalent:*

(i) *$K$ is a linear cosun in $l^p(n)$.*

(ii) *$K$ is the intersection of a family of linear hyperplanes whose $l^2(n)$-normals have at most two nonzero coordinates.*

(iii) *$K$ is the range of a contractive linear projection in $l^p(n)$.*

*It is well known that* (iii) *is equivalent to the condition that $K$ is isometrically isomorphic to some $l^p$-space. This is a special case of a result due to Ando* [1] ($p > 1$) *and Douglas* [13] ($p = 1$), *respectively.*

In $l^\infty(n)$ the class of cosuns is much larger than in $l^p(n)$, $1 < p < \infty$, $p \neq 2$, even in the linear and convex case. The exact characterization of cosuns in $l^\infty(n)$ is left to another paper.

## REFERENCES

1. T. ANDO, Contractive projections in $L_p$ spaces, *Pacific J. Math.* **17** (1966), 391–405.
2. B. BEAUZAMY, Projections contractantes dans les espaces de Banach, *Bull. Sci. Math. (2)* **102** (1978), 43–47.
3. B. BEAUZAMY AND B. MAUREY, Points minimaux et ensembles optimaux dans les espaces de Banach, *J. Funct. Anal.* **24** (1977), 107–139.
4. H. BERENS, Über die beste Approximation im $\mathbb{R}^n$, *Arch. Math. (Basel)* **39** (1982), 376–382.
5. H. BERENS AND U. WESTPHAL, On the best co-approximation in a Hilbert space, *in* "Quantitative Approximation" (R. A. DeVore and K. Scherer, Eds.), pp. 7–10, Academic Press, New York, 1980.
6. L. M. BLUMENTHAL, "Theory and Applications of Distance Geometry," Oxford Univ. Press (Clarendon), London/New York, 1953.

7. F. BOHNENBLUST, Subspaces of $l_{p,n}$ spaces, *Amer. J. Math.* **63** (1941), 64–72.
8. V. G. BOLTYANSKII AND P. S. SOLTAN, Combinatorial geometry and convexity classes, *Russian Math. Surveys* **33** (1978), 1–45.
9. F. E. BROWDER, Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces, *Arch. Rational Mech. Anal.* **24** (1967), 82–90.
10. R. E. BRUCK, Nonexpansive projections on subsets of Banach spaces, *Pacific J. Math.* **47** (1973), 341–355.
11. D. G. DEFIGUEIREDO AND L. A. KARLOVITZ, The extension of contractions and the intersection of balls in Banach spaces, *J. Funct. Anal.* **11** (1972), 168–178.
12. W. G. DOTSON, JR., Fixed points of quasi-nonexpansive mappings, *J. Austral. Math. Soc. Ser. A* **13** (1972), 167–170.
13. R. G. DOUGLAS, Contractive projections on an $L_1$ space, *Pacific J. Math.* **15** (1965), 443–462.
14. C. FRANCHETTI AND M. FURI, Some characteristic properties of real Hilbert spaces, *Rev. Roumaine Math. Pures Appl.* **17** (1972), 1045–1048.
15. P. M. GRUBER, Fixpunktmengen von Kontraktionen in endlich-dimensionalen Räumen, *Geom. Dedicata* **4** (1975), 173–198.
16. L. HETZELT, "Über die beste Coapproximation im $\mathbb{R}^n$," Dissertation, University of Erlangen-Nürnberg, November 1981.
17. L. HETZELT, On suns and cosuns in finite dimensional normed real vector spaces, *Acta Math. Hungar.* **45** (1985), 53–68.
18. J. M. LASRY AND R. ROBERT, "Analyse non linéaire multivoque," Cahiers de Mathématiques de la Décision, No. 7611, Université Paris IX, Dauphine.
19. P. L. PAPINI, Approximation and strong approximation in normed spaces via tangent functionals, *J. Approx. Theory* **22** (1978), 111–118.
20. P. L. PAPINI AND I. SINGER, Best coapproximation in normed linear spaces, *Monatsh. Math.* **88** (1979), 27–44.
21. L. P. VLASOV, Approximative properties of sets in normed linear spaces, *Russian Math. Surveys* **28** (1973), 1–66.
22. U. WESTPHAL, Über Existenz- und Eindeutigkeitsmengen bei der besten Ko-Approximation, *in* "Functional Analysis and Approximation" (P. L. Butzer, B. Sz.-Nagy, and E. Görlich, Eds.), pp. 255–264, Birkhäuser, Basel, 1981.